

Near action degeneracy of periodic orbits in systems with non-conventional time reversal

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Recently, Sieber and Richter calculated semiclassically a first off-diagonal contribution to the orthogonal form factor for a billiard on a surface of constant negative curvature by considering orbit pairs having almost the same action. For a generalization of this derivation to systems invariant under non-conventional time reversal symmetry, which also belong to the orthogonal symmetry class, we show in this paper that it is necessary to redefine the configuration space in an appropriate way.

I. INTRODUCTION

The form factor $K(\tau)$ is defined as the Fourier transformation of the two point energy correlation function of the quantum system where τ is time measured in units of the Heisenberg time T_H . If the Gutzwiller trace formula is used for the density of energy levels the form factor becomes a double sum over the classical periodic orbits γ with the period $T = \tau T_H$:

$$K(\tau) = \lim_{\hbar \rightarrow 0} \frac{1}{T_H} \left\langle \sum_{\gamma, \gamma'} A_\gamma A_{\gamma'}^* e^{i(S_\gamma - S_{\gamma'})/\hbar} \delta\left(T - \frac{T_\gamma + T_{\gamma'}}{2}\right) \right\rangle. \quad (1)$$

Here, A_γ are the stability coefficients of the orbits; the angle brackets signify average over a small interval of time [1].

Only pairs of orbits whose action difference is not large compared with \hbar can make a contribution surviving the time averaging. The diagonal approximation [2] takes into account the diagonal terms $\gamma = \gamma'$ and pairs of mutually time reversed orbits in the case that the dynamics of the system is invariant with respect to time reversal (TR). The result of the diagonal approximation is then

$$K_{\text{diag}}(\tau) = 2\tau. \quad (2)$$

which is to be compared with the form factor of the Gaussian ensemble of random orthogonal matrices for $0 < \tau < 1$,

$$K^{GOE}(\tau) = 2\tau - \tau \log(1 + 2\tau). \quad (3)$$

Obviously, the diagonal approximation reproduces only the first term of the Taylor expansion of the random matrix form factor. It may be expected that taking into account less obvious pairs of periodic orbits with small action difference higher terms of the random matrix form factor (3) will be recovered, in line with the conjecture stated by Bohigas, Giannoni and Schmidt [4]. In the theory of disordered systems the so-called weak diagonal approximation was used by Smith *et al* [5] who stressed the importance of pairs of trajectories with the multiple-loop topology. The pairs of orbits whose contribution seems to be responsible for the higher order terms of the form factor in the case of clean chaos have been discovered only recently by Sieber and Richter [3].

The Sieber-Richter pair of orbits is schematically shown in Fig.1. One of its members contains a self-intersection with a small crossing angle ϵ and consists of two loops, one of which is passed clockwise and another one counterclockwise. Playing with small deformations of such an orbit it can be demonstrated that there exists a partner periodic orbit which is almost everywhere exponentially close to the original one; however, at one place in configuration space the partner orbit has an avoided crossing at the place of a self-intersection.

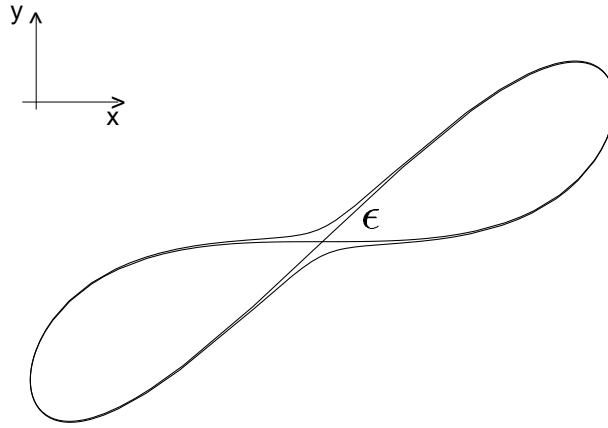


FIG. 1. Sieber-Richter pairs in configuration space

The partner can be regarded as the same two-loop construction as the orbit with crossing, however, now both loops are passed in the same sense of traversal. The time reversal symmetry is essential for the existence of the partner; otherwise retracing the motion along one of the loops would not be allowed. The difference of actions of the orbit and its partner is

$$\Delta S \sim \frac{p^2 \epsilon^2}{2m\lambda} \quad (4)$$

where λ is the Lyapunov constant of the orbit. For small ϵ it can be of the order of \hbar .

The stumbling block is to evaluate the number of the Sieber-Richter pairs and to sum over their contributions. Up to now it has been done only for the billiards in the space of constant negative curvature where all orbits have the same Lyapunov constant and the Maslov index equals zero. Contribution of all pairs Fig.1 have been found to be $-2\tau^2$ [3] which coincides with the term of the order τ^2 in the expansion of the random matrix form factor (3).

One of the open problems in the Sieber-Richter theory is connected with the so-called non-conventional time reversal (NCTR) symmetry. It is often encountered in systems in external magnetic field provided that there exists a suitable spatial symmetry [1]. Their Hamiltonian is invariant under the conventional time reversal combined with an appropriate spatial symmetry operation. Each periodic orbit of a NCTR-symmetric system has a twin with the same action. However, unlike the conventional TR dynamics, the twin is not the same orbit retraced backwards.

The statistical properties of the energy levels of the systems with NCTR and the conventional TR symmetry are identical. In particular, the form factor of the systems with NCTR is given by (3). Therefore, the diagonal approximation is not sufficient, and pairs of orbits with very close action must also exist and contribute to $K(\tau)$.

At the first glance it seems that the Sieber-Richter arguments collapse in the case of NCTR. Even a very weak magnetic field destroys the closeness of action in the pair because of the totally different magnetic flux (due to the fact that one of the loops is passed in the opposite sense by the members of the pair). In a stronger field whose impact on the trajectory cannot be neglected the pairs in Fig. 1 simply cannot exist because passing the same loop in the opposite direction would contradict the equations of motion. The spatial symmetry implied by NCTR does not help. Since the correct semiclassical explanation of the form factor (3) must be essentially the same, be it systems with TR or NCTR, the inability to produce the contributing pairs of orbits in the NCTR-symmetric systems might compromise the whole Sieber-Richter theory. Below we show that these apprehensions are groundless, and the case of NCTR does not present any new difficulties.

Consider a two dimensional motion in the orthogonal uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ and suppose that the potential energy has the symmetry

$$V(x, -y) = V(x, y) . \quad (5)$$

We shall use the gauge $A_x = -By$, $A_y = A_z = 0$. Then the classical Hamiltonian of the system

$$H = \frac{1}{2m} \left(p_x + \frac{eBy}{c} \right)^2 + \frac{p_y^2}{2m} + V(x, y) \quad (6)$$

will be invariant with respect to NCTR consisting of the conventional time reversal TR (changing the sign of the canonical momenta) followed by the reflection in the x axis of the plane (replacement $y \rightarrow -y$, $p_y \rightarrow -p_y$). An arbitrary periodic orbit of our system with the trajectory

$$x = x(t), \quad y = y(t), \quad (7)$$

and the momentum $p_y(t) = m\dot{y}(t)$ along y has a NCTR twin,

$$\tilde{x}(t) = x(-t), \quad \tilde{y}(t) = -y(-t), \quad \tilde{p}_y(t) = p_y(-t). \quad (8)$$

The twin has the same magnetic flux and action. Its trajectory is obtained from the original one by reflection in the x axis while the sense of traversal on both orbits is the same. Let us canonically transform the variables as

$$x' = x, \quad y' = p_y, \quad p'_x = p_x, \quad p'_y = -y \quad (9)$$

and consider how our two orbits project on the new configuration space $x'y'$. The original orbit will be described by the equations

$$x'(t) = x(t), \quad y'(t) = p_y(t) \quad (10)$$

with x, p_y given by Eqs. (7). Its NCTR twin obeys

$$\tilde{x}'(t) = x(-t), \quad \tilde{y}'(t) = p_y(-t) \quad (11)$$

The only difference between (10) and (11) is the change of sign of t . Consequently, in the xp_y plane the two orbits are depicted by the same closed curve traversed in opposite directions: NCTR acts on a periodic orbit in the coordinates xp_y exactly like the usual time reversal in the ordinary configuration space.

Now it is easy to see that in the case of NCTR the Sieber-Richter arguments for the existence of the two-loop pairs of orbits Fig.1 remain fully valid. However, these pairs may only exist in the xp_y plane (Fig.2a) which is the configuration plane in the new coordinates. Only in this projection of the phase space to a two-dimensional submanifold switching the sense of traversal of a loop caused by the replacement of crossing by an avoided crossing is compatible with the equations of motion. Therefore, only in the xp_y plane pairs as depicted in Fig.1 exist and have close actions in strong magnetic fields. Evaluation of the number of Sieber-Richter pairs for the system with NCTR and their contribution to the form factor therefore does not lead to new difficulties as compared to the problem of conventional TR-invariant systems.

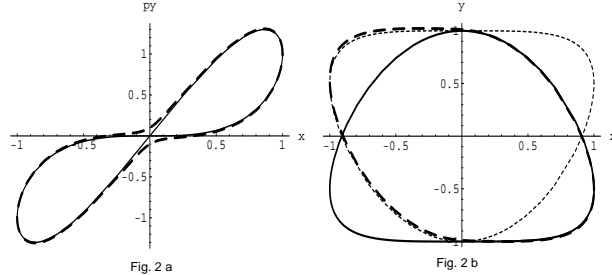


FIG. 2. Sieber-Richter pairs in a NCTR system

It is instructive to see how the pairs Fig. 2a look like when the respective motion is projected onto the usual configuration space xy (Fig.2b). They have little in common with the double-loop Sieber-Richterpairs of Fig.1. However, the general idea of building a new orbit with practically the same action by gently reconnecting parts of the original orbit and its NCTR twin is still obvious. Depending on the projection chosen, the criterion for finding Sieber-Richter pairs changes. In systems with NCTR one may either look for the two loop orbits with small opening angle in the xp_y plane or search for the orbits like in Figure 2b. The latter may be preferable for systems like billiards in the magnetic field whose trajectory in the xp_y plane is discontinuous.

It may be somewhat puzzling that a jump in representation is needed for the Sieber-Richter treatment when the magnetic field is switched on: instead of xy space we must shift to xp_y . However such a jump is only natural in view of the change of the universality class of the dynamics. A chaotic system with the NCTR symmetry belongs to the Gaussian orthogonal ensemble only in the presence of the magnetic field. When the field is switched off the spectrum splits into two independent subspectra (even and odd with respect to $y \rightarrow -y$). Superposition of two such spectra whose levels may cross each other creates a specific ensemble obviously different from GOE, usually called $\text{GOE} \times \text{GOE}$.

The systems with the plane of symmetry in the uniform magnetic field constitute the most important but not the only example of NCTR. Consider e.g. a two-dimensional system with a center of symmetry

$$V(-x, -y) = V(x, y) \quad (12)$$

in an extremely non-uniform magnetic field

$$\mathbf{B} = y\mathbf{e}_z \quad (13)$$

In the gauge $A_x = -By^2/2$, $A_y = A_z = 0$ the Hamiltonian of the system is invariant with respect to the NCTR composed of the time and spatial inversion which means that the momenta are unchanged. The partner obtained from a periodic orbit by this symmetry operation coincides with the original orbit if we draw it in the plane of momenta $p_x p_y$, hence it is in this plane that the Sieber-Richter pairs are described by small intersection angle ϵ .

To summarize, the Sieber-Richter double loop pairs Fig. 1 or Fig. 2a may be observed only in the configuration space whose generalized coordinates are unchanged by the particular NCTR. A periodic orbit and its NCTR twin are depicted in this space by the same curve traversed in opposite direction. This may be the usual coordinate space xy in systems with conventional time reversal symmetry, the xp_y space in systems with the NCTR $t \rightarrow -t, y \rightarrow -y$, and the $p_x p_y$ space if NCTR is described by $t \rightarrow -t, x \rightarrow -x, y \rightarrow -y$. An attempt to break out of this symmetry-dictated space by a canonical transformation mixing the coordinates and momenta will immediately strip the Sieber-Richter pairs of their double-loop, intersection/avoided-crossing appearance. However, using appropriate criteria to describe the close action partners can be recognized in principle in any two dimensional projection of the phase space.

Appendix

Calculations in [3] are based on the geometric evaluation of the shifts of momenta using the fact that velocities and momenta in the conventional coordinate space are practically identical. After our canonical transform the connection between the new “velocities” and the momenta becomes more complicated. In particular the direction of the momentum is no longer tangent to the trajectory in the new configuration space. That means that the Sieber-Richter result has to be rederived.

We assume the existence of a self-crossing periodic orbit with a small opening angle like the one shown in Fig. 1, 2a. A coordinate frame is introduced with its origin at the point of crossing, and the x' axis along the bisector of the small angle. Consider the Poincaré section at $x' = 0$ with the coordinates y' and p'_y on the crossing plane. The true Poincaré map (Fig.3) is obtained when passages of the $x' = 0$ plane with a certain sign of \dot{x}' , say, $\dot{x}' > 0$ are marked. The self-crossing orbit will then be depicted by a periodic point O on the p'_y axis with p'_y positive and small. The self-crossing TR (NCTR) twin of this orbit will produce another periodic point O' symmetrical with respect to the y' axis: $y' = 0, p'_y < 0$.

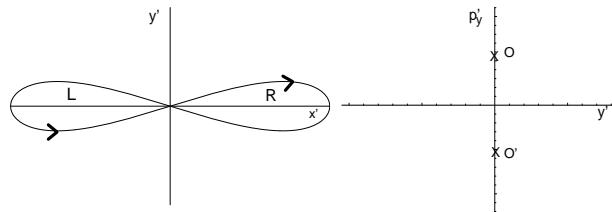


FIG. 3. Poincaré map of the Sieber-Richter Orbit

We shall concentrate, however, on the submaps R, L of the Poincaré map describing the transform of y', p'_y generated by the right and left loop of the orbit. We shall mark the crossing of the $x' = 0$ plane both for $\dot{x}' > 0$ and $\dot{x}' < 0$; such a break of the rules is needed since R and L' are not true Poincaré maps. We shall also be interested in the TR (NCTR) submaps obtained by passing the loops of the orbit in the direction opposite to Fig. 3; these will be denoted R' and L , respectively. The periodic point O of the total Poincaré map is simultaneously the periodic point of the submaps R and L whereas O' is the periodic point of the submaps L' and R' .

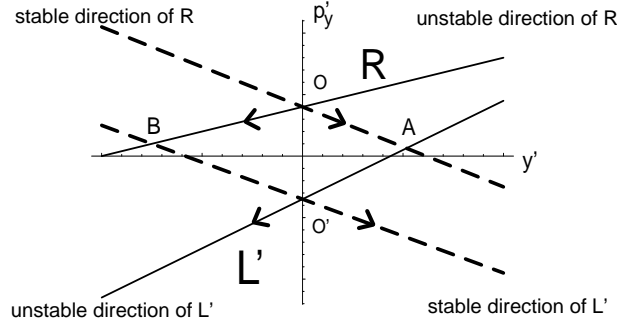


FIG. 4. Stable and instable manifolds of the right submap R and the time-reversed submap L' . A point P in the vicinity of A will be mapped to a point Q in the vicinity of point B . Fine-tuning of the initial point P leads to a periodic orbit with avoided self-crossing.

Each submap can be characterized by its stability matrix connecting the initial and final deflections of the y' coordinate and momentum from the periodic point of the respective submap. We shall need M_R (the stability matrix of the right loop passed as it is shown in Fig. 3) and $M_{L'}$ (the one for the left loop followed in the direction opposite to Fig. 3). The large eigenvalues Λ_R and $\Lambda_{L'}$ of these two matrices can be evaluated as $\sim \exp(\lambda T)$ where λ is the Lyapunov constant and $T = T_R, T_{L'}$ is the period of the respective loop. The periods of the orbits in the sum for the form factor (1) are of the order $\hbar^{-1} \rightarrow \infty$, therefore the larger eigenvalues of the stability matrices are exponentially large whereas the smaller ones ($1/\Lambda$) are exponentially close to zero. The respective eigenvectors determine the unstable and stable directions of each of the submaps.

Consider Fig. 4 where periodic points O, O' of the submaps R, L' and their stable and unstable directions are shown. Let us investigate the application of the submap R to an initial point P chosen in the vicinity of the crossing A of the stable direction of R and unstable direction of L' . We shall represent the initial radius-vector by an expansion in powers of Λ_R^{-1} ,

$$\mathbf{r}_P = \mathbf{e}_s^R \left(l_{OA} + \frac{c_1}{\Lambda_R} + \dots \right) + \mathbf{e}_u^R \left(\frac{l_{OB}}{\Lambda_R} + \frac{d_2}{\Lambda_R^2} + \dots \right) . \quad (14)$$

Here $\mathbf{e}_s^R, \mathbf{e}_u^R$ are the eigenvectors of the stability matrix of R along the stable and unstable directions, l_{OA} and l_{OB} are distances from A and B to the periodic point O (B is the crossing of the unstable direction of R and stable direction of L'); the coefficients c_1, d_2, \dots are so far undetermined.

After the loop R has been completed the point P will be mapped to the point Q obtained by squeezing along the stable and stretching along the unstable direction with the coefficient Λ_R :

$$\mathbf{r}_Q = \mathbf{e}_s^R \left(\frac{l_{OA}}{\Lambda_R} + \frac{c_1}{\Lambda_R^2} + \dots \right) + \mathbf{e}_u^R \left(l_{OB} + \frac{d_2}{\Lambda_R} + \dots \right) . \quad (15)$$

It is seen that Q is infinitely close to the crossing point B of the unstable direction of R and the stable direction of L' ; thus the distance of Q from the stable manifold of L' is exponentially small and depends on the coefficients c_1, d_2, \dots

Now consider the loop L' taking Q as its initial point. The loop will practically annihilate the stable component $\mathbf{e}_u^R l_{OB}$ and place the final point somewhere on its unstable manifold. The exact position of the final point on the unstable manifold of L' depends on c_1, d_2, \dots ; these can be finetuned so that the final point will coincide with the initial point P of the loop R .¹ But that would mean that P is a periodic point of the composition of the submaps R and L' . It corresponds to a new periodic orbit composed of the deformed loops R and L' . The new orbit crosses the “true” Poincaré map at the point infinitely close to the crossing of the stable manifold of R and unstable manifold of L' . The TR (NCTR) twin of the new periodic orbit can be found by considering the sequence of the submaps R' and L . The new orbit and its twin are of course the Sieber-Richter partners with avoided crossing of the orbit with self-intersection.

¹More accurately, demanding that the final point of the second loop coincides with the initial point P of the first loop we obtain a set of equations for consecutive definition of $c_1, d_2, c_2, d_3, \dots$. The coefficients in the expansion will not be growing and convergence for \mathbf{r}_P guaranteed provided $\Lambda_R < \Lambda_L$ which can be assumed without loss of generality.

As can be seen we have not used the connection between the velocities and momenta of the usual configuration space. Note also that the effect of the pair formation seems to remain in force even when the nonlinear corrections to the submaps R, L' are taken into account, with their stable and unstable manifolds depicted by curves rather than straight lines, as long as the stable and unstable manifold intersect only once. For small angle ϵ , this condition should be fulfilled.

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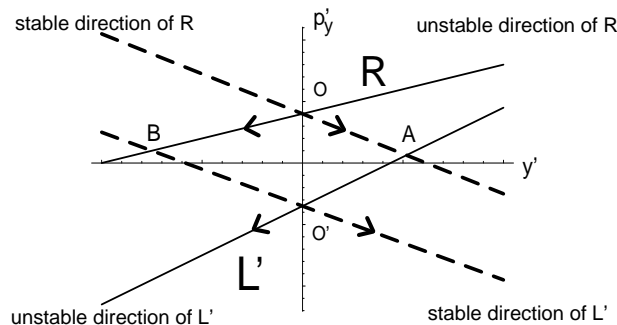


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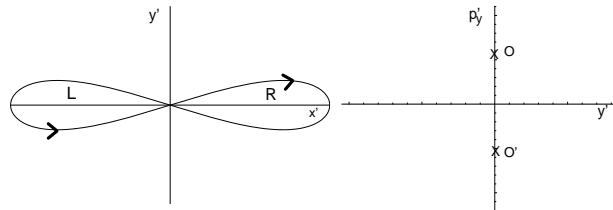


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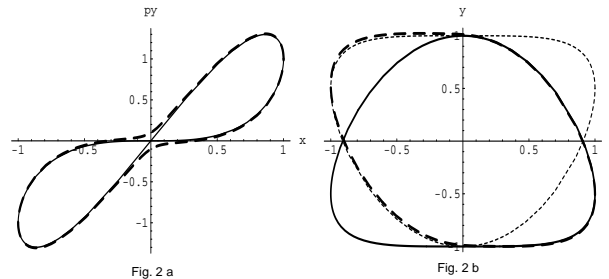


FIG. 2. Sieber-Richter pairs in a NCTR system

It is instructive to see how the pairs Fig. 2a look like when the respective motion is projected onto the usual configuration space xy (Fig.2b). They have little in common with the double-loop Sieber-Richterpairs of Fig.1. However, the general idea of building a new orbit with practically the same action by gently reconnecting parts of the original orbit and its NCTR twin is still obvious. Depending on the projection chosen, the criterion for finding Sieber-Richter pairs changes. In systems with NCTR one may either look for the two loop orbits with small opening angle in the xp_y plane or search for the orbits like in Figure 2b. The latter may be preferable for systems like billiards in the magnetic field whose trajectory in the xp_y plane is discontinuous.

It may be somewhat puzzling that a jump in representation is needed for the Sieber-Richter treatment when the magnetic field is switched on: instead of xy space we must shift to xp_y . However such a jump is only natural in view of the change of the universality class of the dynamics. A chaotic system with the NCTR symmetry belongs to the Gaussian orthogonal ensemble only in the presence of the magnetic field. When the field is switched off the spectrum splits into two independent subspectra (even and odd with respect to $y \rightarrow -y$). Superposition of two such spectra whose levels may cross each other creates a specific ensemble obviously different from GOE, usually called $\text{GOE} \times \text{GOE}$.

The systems with the plane of symmetry in the uniform magnetic field constitute the most important but not the only example of NCTR. Consider e.g. a two-dimensional system with a center of symmetry

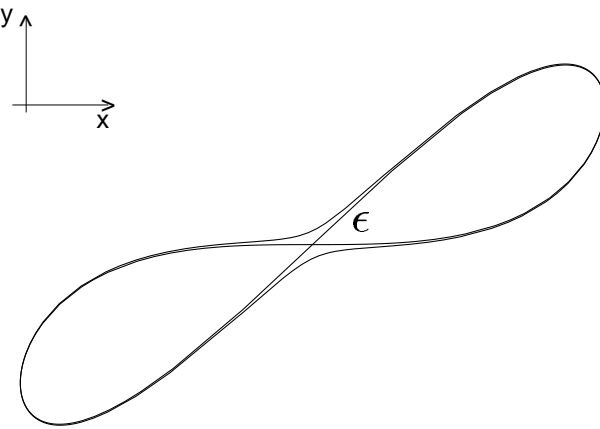


FIG. 1. Sieber-Richter pairs in configuration space

The partner can be regarded as the same two-loop construction as the orbit with crossing, however, now both loops are passed in the same sense of traversal. The time reversal symmetry is essential for the existence of the partner; otherwise retracing the motion along one of the loops would not be allowed. The difference of actions of the orbit and its partner is

$$\Delta S \sim \frac{p^2 \epsilon^2}{2m\lambda} \quad (4)$$

where λ is the Lyapunov constant of the orbit. For small ϵ it can be of the order of \hbar .

The stumbling block is to evaluate the number of the Sieber-Richter pairs and to sum over their contributions. Up to now it has been done only for the billiards in the space of constant negative curvature where all orbits have the same Lyapunov constant and the Maslov index equals zero. Contribution of all pairs Fig.1 have been found to be $-2\tau^2$ [3] which coincides with the term of the order τ^2 in the expansion of the random matrix form factor (3).

One of the open problems in the Sieber-Richter theory is connected with the so-called non-conventional time reversal (NCTR) symmetry. It is often encountered in systems in external magnetic field provided that there exists a suitable spatial symmetry [1]. Their Hamiltonian is invariant under the conventional time reversal combined with an appropriate spatial symmetry operation. Each periodic orbit of a NCTR-symmetric system has a twin with the same action. However, unlike the conventional TR dynamics, the twin is not the same orbit retraced backwards.

The statistical properties of the energy levels of the systems with NCTR and the conventional TR symmetry are identical. In particular, the form factor of the systems with NCTR is given by (3). Therefore, the diagonal approximation is not sufficient, and pairs of orbits with very close action must also exist and contribute to $K(\tau)$.

At the first glance it seems that the Sieber-Richter arguments collapse in the case of NCTR. Even a very weak magnetic field destroys the closeness of action in the pair because of the totally different magnetic flux (due to the fact that one of the loops is passed in the opposite sense by the members of the pair). In a stronger field whose impact on the trajectory cannot be neglected the pairs in Fig. 1 simply cannot exist because passing the same loop in the opposite direction would contradict the equations of motion. The spatial symmetry implied by NCTR does not help. Since the correct semiclassical explanation of the form factor (3) must be essentially the same, be it systems with TR or NCTR, the inability to produce the contributing pairs of orbits in the NCTR-symmetric systems might compromise the whole Sieber-Richter theory. Below we show that these apprehensions are groundless, and the case of NCTR does not present any new difficulties.

Consider a two dimensional motion in the orthogonal uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ and suppose that the potential energy has the symmetry

$$V(x, -y) = V(x, y) . \quad (5)$$

We shall use the gauge $A_x = -By$, $A_y = A_z = 0$. Then the classical Hamiltonian of the system

$$H = \frac{1}{2m} \left(p_x + \frac{eBy}{c} \right)^2 + \frac{p_y^2}{2m} + V(x, y) \quad (6)$$

will be invariant with respect to NCTR consisting of the conventional time reversal TR (changing the sign of the canonical momenta) followed by the reflection in the x axis of the plane (replacement $y \rightarrow -y$, $p_y \rightarrow -p_y$). An arbitrary periodic orbit of our system with the trajectory

Near action degeneracy of periodic orbits in systems with non-conventional time reversal

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Recently, Sieber and Richter calculated semiclassically a first off-diagonal contribution to the orthogonal form factor for a billiard on a surface of constant negative curvature by considering orbit pairs having almost the same action. For a generalization of this derivation to systems invariant under non-conventional time reversal symmetry, which also belong to the orthogonal symmetry class, we show in this paper that it is necessary to redefine the configuration space in an appropriate way.

I. INTRODUCTION

The form factor $K(\tau)$ is defined as the Fourier transformation of the two point energy correlation function of the quantum system where τ is time measured in units of the Heisenberg time T_H . If the Gutzwiller trace formula is used for the density of energy levels the form factor becomes a double sum over the classical periodic orbits γ with the period $T = \tau T_H$:

$$K(\tau) = \lim_{\hbar \rightarrow 0} \frac{1}{T_H} \left\langle \sum_{\gamma, \gamma'} A_\gamma A_{\gamma'}^* e^{i(S_\gamma - S_{\gamma'})/\hbar} \delta \left(T - \frac{T_\gamma + T_{\gamma'}}{2} \right) \right\rangle. \quad (1)$$

Here, A_γ are the stability coefficients of the orbits; the angle brackets signify average over a small interval of time [1].

Only pairs of orbits whose action difference is not large compared with \hbar can make a contribution surviving the time averaging. The diagonal approximation [2] takes into account the diagonal terms $\gamma = \gamma'$ and pairs of mutually time reversed orbits in the case that the dynamics of the system is invariant with respect to time reversal (TR). The result of the diagonal approximation is then

$$K_{\text{diag}}(\tau) = 2\tau. \quad (2)$$

which is to be compared with the form factor of the Gaussian ensemble of random orthogonal matrices for $0 < \tau < 1$,

$$K^{GOE}(\tau) = 2\tau - \tau \log(1 + 2\tau). \quad (3)$$

Obviously, the diagonal approximation reproduces only the first term of the Taylor expansion of the random matrix form factor. It may be expected that taking into account less obvious pairs of periodic orbits with small action difference higher terms of the random matrix form factor (3) will be recovered, in line with the conjecture stated by Bohigas, Giannoni and Schmidt [4]. In the theory of disordered systems the so-called weak diagonal approximation was used by Smith *et al* [5] who stressed the importance of pairs of trajectories with the multiple-loop topology. The pairs of orbits whose contribution seems to be responsible for the higher order terms of the form factor in the case of clean chaos have been discovered only recently by Sieber and Richter [3].

The Sieber-Richter pair of orbits is schematically shown in Fig.1. One of its members contains a self-intersection with a small crossing angle ϵ and consists of two loops, one of which is passed clockwise and another one counterclockwise. Playing with small deformations of such an orbit it can be demonstrated that there exists a partner periodic orbit which is almost everywhere exponentially close to the original one; however, at one place in configuration space the partner orbit has an avoided crossing at the place of a self-intersection.